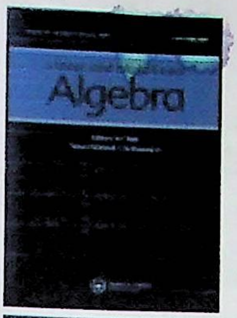

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# Matrices over non-commutative rings as sums of powers

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**ABSTRACT**

Let $R$ be non-commutative ring with unity and $n \geq p \geq 2$, $p$ prime. In this paper, we prove that an $n \times n$ matrix over $R$ is the sum of $p$th powers if and only if its trace can be written as a sum of $p$th powers and commutators modulo $pR$. This extends the results of L. N. Vaserstein ($p = 2$) and S. A. Katre, Kshipra Wadikar ($p = 3$). We also obtain necessary and sufficient conditions for a matrix over $R$ to be written as a sum of fourth powers when $n \geq 2$.

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## 1. Introduction

Carlitz showed as a solution to a problem proposed in Canadian Mathematical Bulletin that every $2 \times 2$ integer matrix is a sum of at most 3 squares (see [1]). Initial work related to integer matrices and matrices over commutative rings as sums of squares can be found in [2, 3]. Wadikar and Katre [4] proved that every integer matrix is a sum of four cubes. Richman [5] studied Waring's problem for matrices over commutative rings as sums of $k$th powers. Katre and Garge [6] gave generalized trace condition for a matrix over a commutative ring to be a sum of $k$th powers.

All our rings are associative. By a non-commutative ring, we mean a ring with unity which is not necessarily commutative. In this paper, $R$ will be a non-commutative ring, and $M_n(R)$ will denote the ring of $n \times n$ matrices over $R$. For a non-commutative ring $R$, Vaserstein proved that a matrix of size $n \geq 2$ over $R$ is a sum of squares if and only if its trace is a sum of squares modulo $2R$ (see [7]). Recently, Katre and Wadikar proved that a matrix of size $n \geq 2$ over $R$ is a sum of cubes if and only if its trace is a sum of cubes and commutators modulo $3R$ (see [8]). In this paper, in the context of Waring's problem for matrices, we obtain such a result for $p$th powers when $n \geq p \geq 2$, $p$ prime. We also obtain an analogue of this result for fourth powers for $n \geq 2$. For both these results, we use the

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following general trace condition for a matrix over a non-commutative ring to be a sum of $k$th powers ([8], Theorem 3.2).

**Theorem A (Katre, Wadikar):** Let $n, k \geq 2$ be integers and $A \in M_n(R)$. $A$ is a sum of $k$th powers of matrices in $M_n(R)$ if and only if trace$(A)$ is a sum of traces of $k$th powers of matrices in $M_n(R)$.

**Notations:** $E_{ij}$: The $n \times n$ matrix whose $(i,j)$th entry is 1 and other entries are 0. $E'_{ij}$: The $p \times p$ matrix whose $(i,j)$th entry is 1 and other entries are 0. $C(a_1, a_2, \ldots, a_k) = a_1a_2\cdots a_k + a_2a_3\cdots a_ka_1 + \cdots + a_ka_1a_2\cdots a_{k-1}$, where $a_1, a_2, \ldots, a_k \in R$, is called a cyclic sum of length $k$. $[x,y] = xy - yx$ is called the commutator of $x$ and $y$. Note that $-C(a_1, a_2, \ldots, a_k) = C(-a_1, a_2, \ldots, a_k)$ is a cyclic sum and $-[x,y] = [-x,y]$ commutator.

## 2. Matrices as sums of $p$th powers

**Proposition 2.1:** *For $a_1, a_2, \ldots, a_k \in R$, the cyclic sum $C(a_1, a_2, \ldots, a_k)$ is a sum of commutators modulo $kR$.*

***Proof:*** Observe that $a_1a_2\cdots a_k + a_2a_3\cdots a_ka_1 + \cdots + a_ka_1a_2a_3\cdots a_{k-1} = (a_2a_3\cdots a_k)a_1 - a_1(a_2a_3\cdots a_k) + (a_3a_4\cdots a_k)(a_1a_2) - (a_1a_2)(a_3a_4\cdots a_k) + \cdots + a_k(a_1a_2\cdots a_{k-1}) - (a_1a_2\cdots a_{k-1})a_k + ka_1a_2\cdots a_k = [a_2a_3\cdots a_k, a_1] + [a_3a_4\cdots a_k, a_1a_2] + \cdots + [a_k, a_1a_2\cdots a_{k-1}] + ka_1a_2\cdots a_k$.

We also consider the action of the cyclic group generated by $\sigma = (1, 2, \ldots, k) \in S_k$, the permutation group on $k$ symbols, on the set of $k$-tuples of elements of a set. This action is defined by $\sigma(a_1, a_2, \ldots, a_k) = (a_2, a_3, \ldots, a_k, a_1)$. If $k = p$ is a prime, then, since the number of elements in the orbit of any $p$-tuple divides $p$ = order of the group $\langle\sigma\rangle$, the orbit has 1 or $p$ elements. Hence if at least two of $a_1, a_2, \ldots, a_p$ are unequal, the orbit has exactly $p$ elements. ■

**Proposition 2.2:** *If $R$ is a non-commutative ring and $n \geq p \geq 2$, $p$ prime, then for $A \in M_n(R)$, trace$(A^p)$ is the sum of $p$th powers of diagonal elements of $A$ and cyclic sums $C(a_1, a_2, \ldots, a_p)$ with $a_1, a_2, \ldots, a_p \in R$.*

***Proof:*** If $A = (a_{ij})$, then trace$(A^p) = \sum_{1\leq j_i\leq n} a_{j_1j_2}a_{j_2j_3}\cdots a_{j_pj_1} = \sum_{i=1}^{n} a_{ii}^p + \sum_{(j_1,j_2,\ldots,j_p)\in B} a_{j_1j_2}a_{j_2j_3}\cdots a_{j_pj_1}$ where $B$ is the set of all $(j_1, j_2, \ldots, j_p)$ for which at least two of $j_1, j_2, \ldots, j_p$ are unequal. Since $p$ is a prime, for $(j_1, j_2, \ldots, j_p) \in B$, the orbit of $(j_1, j_2, \ldots, j_p)$ under the cyclic change, i.e. the action of the cycle $\sigma = (1, 2, \ldots, p) \in S_p$, has $p$ elements. Thus, there are $p$ distinct $p$-tuples obtained from cyclic changes in $(j_1, j_2, \ldots, j_p)$, and they together give rise to $C(a_{j_1j_2}, a_{j_2j_3}, \ldots, a_{j_pj_1})$. All such cyclic sums corresponding to different orbits give rise to the second sum. ■

**Theorem 1:** *Let $n \geq p \geq 2$, $p$ prime, be integers. Let $T_p = T_{p,n}$ be the set of those elements of $R$ that can be expressed as sums of traces of $p$th powers of $n \times n$ matrices over $R$.*

(i) *For $a, a_1, a_2, \ldots, a_p \in R$, the cyclic sum $C(a_1, a_2, \ldots, a_p) \in T_p$. Also $pa \in T_p$, $a^p \in T_p$.*

(ii) $T_p = \{\sum_{(a_1,a_2,\ldots,a_p)\in S} C(a_1,a_2,\ldots,a_p) + \sum_{j=1}^{l} c_j^p \mid l \geq 1,$ *S is a finite subset of* $R^p, a_i, c_j \in R, 1 \leq i \leq p, 1 \leq j \leq l\}$.

(iii) $T_p = \{\sum_{(a_1,a_2,\ldots,a_p)\in S} C(a_1,a_2,\ldots,a_p) + c^p \mid$ *S is a finite subset of* $R^p, a_i, c \in R, 1 \leq i \leq p\}$.

(iv) $T_p = \{\sum_{j=1}^{q}(a_jb_j - b_ja_j) + \sum_{j=1}^{l} c_j^p + pr \mid a_j, b_j, c_j, r \in R,\ q \geq 1,\ l \geq 1\}$.

(v) $T_p = \{\sum_{j=1}^{q}(a_jb_j - b_ja_j) + c^p + pr \mid a_j, b_j, c, r \in R,\ q \geq 1,\ l \geq 1\}$.

(vi) *A matrix $A \in M_n(R)$ is a sum of pth powers if and only if trace(A) is a sum of pth powers and commutators modulo pR if and only if trace(A) is a sum of a pth power and commutators modulo pR.*

(vii) *Vaserstein* ([7], Theorem 1): *A matrix $A \in M_n(R)$ is a sum of squares if and only if trace(A) is a sum of squares modulo 2R.*

***Proof:*** (i) Let $E'_{ij}$ be the $p \times p$ matrix as in Section 1 and $O_{n-p}$ be the zero matrix of order $n-p$. Let $F = a_1E'_{12} + a_2E'_{23} + \cdots + a_{p-1}E'_{p-1,p} + a_pE'_{p,1}$. Then, as in the proof of Proposition 2.2, $C(a_1,a_2,\ldots,a_p) = \text{trace}(F^p) = \text{trace}(F \oplus O_{n-p})^p$. Hence $C(a_1,a_2,\ldots,a_p) \in T_p$. As $C(a,1,1,\ldots,1) = pa$, we get $pa \in T_p$. Also $a^p = \text{trace}((aE_{11})^p) \in T_p$, $E_{11}$ being as in Section 1.

(ii) By (i), R.H.S. of (ii) $\subseteq T_p$. Conversely, $T_p \subseteq$ R.H.S. of (ii) by Proposition 2.2.

(iii) Let $S$ be a set of representatives of orbits of $p$-tuples of elements of $\{c_1, c_2, \ldots, c_l\} \subseteq R$ and let $S'$ be the set of such representatives in which we have at least two unequal entries. Then, the multinomial theorem for $p$th powers can be written as

$$(c_1 + c_2 + \cdots + c_l)^p = \sum_{(d_1,d_2,\ldots,d_p)\in S} C(d_1,d_2,\ldots,d_p)$$

$$= \sum_{j=1}^{l} c_j^p + \sum_{(d_1,d_2,\ldots,d_p)\in S'} C(d_1,d_2,\ldots,d_p). \qquad (1)$$

Hence $\sum_{j=1}^{l} c_j^p = (c_1 + c_2 + \cdots + c_l)^p - \sum_{(d_1,d_2,\ldots,d_p)\in S'} C(d_1,d_2,\ldots,d_p)$. Now, $-C(d_1,d_2,\ldots,d_p) = C(-d_1,d_2,\ldots,d_p)$. Thus, $\sum_{j=1}^{l} c_j^p = (c_1 + c_2 + \cdots + c_l)^p + \sum_{(d_1,d_2,\ldots,d_p)\in S'} C(-d_1,d_2,\ldots,d_p)$. Hence using (ii), we get (iii).

(iv) From (ii) and Proposition 2.1, $T_p$ is a subset of the set of sums of commutators and $p$th powers modulo $pR$. Conversely, $[a,b] = ab - ba = a.\underbrace{1.1\cdots 1}_{p-2}.b + \underbrace{1.1.\cdots .1}_{p-2}.b.a + \underbrace{1.1\cdots 1}_{p-3}.b.a.1 + \cdots + b.a.\underbrace{1.1\cdots 1}_{p-2} - p.b.a = C(a,1,\ldots,1,b) \in T_p$ by (ii).

(v) Now, using (iii) and Proposition 2.1, $T_p \subseteq$ R.H.S of (v). By (iv), R.H.S. of (v) $\subseteq T_p$.

(vi) This follows from Theorem A using (iv) and (v).

(vii) We have $[a,b] = ab - ba = (a+b)^2 + a^2 + b^2$ modulo $2R$. Hence by (iv) $T_2 = \{\sum_{j=1}^{l} c_j^2 + 2r \mid c_j, r \in R, l \geq 1\}$. So using (vi) for $p = 2$ we get (vii). See also ([8], Theorem 3.9).

■

Note: $T_p = T_{p,n}$ is independent of $n$ for $n \geq p \geq 2$.

**Corollary 2.1 (Richman, [[5], Proposition 4.2]):** *Let $n \geq p \geq 2$, $p$ prime and $R$ be a commutative ring with unity. $A \in M_n(R)$ is a sum of $p$th powers if and only if trace($A$) is a $p$th power modulo $pR$.*

***Proof:*** Since $R$ is a commutative ring with unity, every commutator is zero. Now use (vi) of Theorem 2.1.

In the case of $p$th powers, we required to show in our proof that a cyclic sum $C(a_1, a_2, \ldots, a_p)$ is in $T_p$. For this, we showed that $C(a_1, a_2, \ldots, a_p) = \text{trace}(F^p)$, where $F$ is a $p \times p$ matrix. Because of this our proof required $n \geq p$. We shall see in the next section that for fourth powers we can make use of the four entries in a $2 \times 2$ matrix to show that $C(a, b, c, d) \in T_4$. This will give us a criterion for $A \in M_n(R)$ to be a sum of fourth powers for $n \geq 2$. ■

## 3. Matrices as sums of fourth powers

The following theorem gives a non-commutative version of Theorem 6.3 in [6].

**Theorem 2:** *Let $n \geq 2$ be an integer and let $T_4 = T_{4,n}$ be the set of those elements of $R$ that can be expressed as sums of traces of fourth powers of $n \times n$ matrices over $R$. For $a, b, c, d \in R$, let $C(a, b, c, d) = abcd + bcda + cdab + dabc$ and $D(a, b) = abab + baba$. Then*

(i) *For $a, b, c, d \in R$, $C(a, b, c, d) \in T_4$. Also $4a, a^4, 2a^2, [a, b], D(a, b) \in T_4$.*
(ii) $T_4 = \{\sum_{j=1}^{q} C(a_j, b_j, c_j, d_j) + \sum_{j=1}^{t} D(e_j, f_j) + \sum_{j=1}^{l} g_j^4 | a_j, b_j, c_j, d_j, e_j, f_j, g_j \in R, q, t, l \geq 1\}$.
(iii) $T_4 = \{\sum_{j=1}^{q}(a_j b_j - b_j a_j) + \sum_{j=1}^{l} c_j^4 + 2\sum_{j=1}^{t} d_j^2 + 4r | a_j, b_j, c_j, d_j \in R, q, l, t \geq 1\}$.
(iv) *A matrix $A \in M_n(R)$ is a sum of fourth powers if and only if trace($A$) is a sum of fourth powers and 2(sum of squares) and commutators modulo $4R$.*
(v) *A matrix $A$ in $M_n(R)$ is a sum of fourth powers if and only if* $\text{trace}(A) = x_0^4 + 2x_1^2 + 4x_2 +$ *a sum of commutators where $x_0, x_1, x_2 \in R$.*

***Proof:*** (i) For the $2 \times 2$ matrix $E'_{ij}$, and the zero matrix $O_{n-2}$ of order $n-2$, let, for $a, b, c, d \in R$,

$$N_1 = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \quad N_2 = \begin{pmatrix} a & -b \\ d & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & -b \\ d & c \end{pmatrix},$$
$$N_4 = \begin{pmatrix} 0 & b \\ d & 0 \end{pmatrix}, \quad N_5 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad N_6 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.$$

We have, $\text{trace}\sum_{i=1}^{4} N_i^4 = [a^4 + C(a, a, b, d) + C(b, c, c, d) + D(b, d) + C(a, b, c, d) + c^4] + [a^4 - C(a, a, b, d) + D(b, d)] + [-C(b, c, c, d) + D(b, d) + c^4] + D(b, d) = 2a^4 + 2c^4 + 4D(b, d) + C(a, b, c, d) = \text{trace}N_5^4 + \text{trace}\, N_6^4 + 4D(b, d) + C(a, b, c, d)$. Hence $C(a, b, c, d) = \text{trace}\sum_{i=1}^{6} N_i^4$ modulo $4R$ $= \text{trace}\sum_{i=1}^{6} (N_i \oplus O_{n-2})^4$ modulo $4R$, so $C(a, b, c, d) \in T_4$. Also $C(a, 1, 1, 1) = 4a$, hence $4a \in T_4$. For $E_{11}$ as in Section 1,

$a^4 = \text{trace}(aE_{11})^4 \in T_4$. Also

$$2a^2 = \text{trace}((E'_{12} + aE'_{21}) \oplus O_{n-2})^4 \in T_4.$$

Since $[a,b] = a \cdot 1 \cdot 1 \cdot b + 1 \cdot 1 \cdot b \cdot a + 1 \cdot b \cdot a \cdot 1 + b \cdot 1 \cdot 1 \cdot a - 4ba$, so $[a,b] \in T_4$. Also $D(a,b) = [a, bab] + 2baba$. Now $[a, bab] \in T_4$ as it is a commutator and $2baba = 2(ba)^2$ is in $T_4$. Hence $D(a,b) \in T_4$.

(ii) From (i), $C(a_j, b_j, c_j, d_j) \in T_4$, also $g_j^4 \in T_4$. Thus, every element of R.H.S. of (ii) $\in T_4$. Conversely, for $A \in M_n(R)$, trace of $A^4$ is sum of fourth powers of diagonal entries and entries of the type $C(a,b,c,d)$ and $D(e,f)$, so $T_4 \subseteq$ R.H.S of (ii).

(iii) By (i), $[a,b] \in T_4$. Also by (i) every term in the elements of R.H.S. of (iii)$\in T_4$, so R.H.S. of (iii)$\subseteq T_4$ and conversely by (ii) $T_4 \subseteq$ R.H.S. of (iii).

(iv) A matrix $A \in M_n(R)$ is a sum of fourth powers if and only if trace of A is a sum of traces of fourth powers of matrices in $M_n(R)$ if and only if, by (iii), trace(A) is a sum of fourth powers and 2(sum of squares) and commutators modulo $4R$.

(v) By (iv), $A$ in $M_n(R)$ is sum of fourth powers if and only if trace($A$) is a sum of fourth powers and 2(sum of squares) and sum of commutators modulo $4R$. Now consider $a^4 + b^4 = (a+b)^4 - (a^3b + a^2ba + aba^2 + ba^3) - (ab^2a + b^2a^2 + ba^2b + a^2b^2) - (b^2ab + bab^2 + ab^3 + b^3a) - (baba + abab) = (a+b)^4 -$ cyclic sums $-[b, aba] + 2(ab)^2$. Since every cyclic sum is a sum of commutators modulo $4R$, we get $a^4 + b^4 = (a+b)^4 + 2(ab)^2 +$ a sum of commutators modulo $4R$. Also $a^2 + b^2 = (a+b)^2 + [a,b] + 2ba$. Using this repeatedly, we get the result. ∎

Note: $T_4$ is independent of $n$ for $n \geq 2$.

**Corollary 3.1 (Katre-Garge, [[6], Theorem 6.3]):** *If R is a commutative ring with unity, then A in $M_n(R)$ is a sum of fourth powers if and only if trace(A)$= x_0^4 + 2x_1^2 + 4x_2$ for some $x_0, x_1, x_2 \in R$.*

***Proof:*** Since $R$ is commutative, all commutators are zero, so the result follows from (v) of Theorem 2. ∎

We note the following relation between trace($M^q$) and trace $M$, for $q$ prime.

**Proposition 3.1:** *Let R be a ring, q prime. For $M \in M_n(R)$, trace $M^q =$ (trace M)$^q$ + a sum of commutators modulo qR.*

***Proof:*** Let $M = (a_{ij})$, then trace$(M^q) = \sum_{1 \leq j_1, j_2, \ldots, j_q \leq n} a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_q j_1}$. If all the $j_i$ are equal, we get a $q$th power and if at least two of $j_1, j_2, \ldots, j_q$ are unequal, $q$ being a prime, there are $q$ distinct $q$-tuples obtained from $j_1, j_2, \ldots, j_q$ by a cyclic change. Thus, trace$(M^q) = a_{11}^q + a_{22}^q + \cdots + a_{qq}^q +$ cyclic sums $= (a_{11} + a_{22} + \cdots + a_{qq})^q +$ a sum of commutators modulo $qR$ by (1) and Proposition 2.1. Hence trace$(M^q) =$ (trace$M)^q +$ sum of commutators modulo $qR$. ∎

**Lemma 3.1:** *For $\alpha \in R$ the following are equivalent:*

(i) *$\alpha$ is a sum of cubes and commutators modulo $3R$.*
(ii) *$\alpha$ is a sum of a cube and commutators modulo $3R$.*

***Proof:*** (ii) implies (i) is clear.

By Equation (1), for $p = 3$, $\sum_{j=1}^{l} c_j^3 = (\sum_{j=1}^{l} c_j)^3 - \sum_{(d_1,d_2,d_3)\in S'} C(d_1,d_2,d_3)$. Now $-C(d_1,d_2,d_3) = C(-d_1,d_2,d_3)$, hence the result follows from Proposition 2.1. ■

Using this, we get the following corollary.

**Corollary 3.2:** *For $n \geq 2$, a matrix $A$ in $M_n(R)$ is a sum of cubes if and only if trace $(A)$ is a sum of a cube and commutators modulo $3R$.*

***Proof:*** The proof follows from ([8], Theorem 3.13 (iv)) and Lemma 3.1. ■

**Remark 3.1:** The results in Theorem 2.1 and Corollary 2.1 are for $n \times n$ matrices as sums of $p$th powers, $p$ prime, when $n \geq p \geq 2$. The problem remains for $p > n \geq 2$, i.e. we expect to get the result for all $n \geq 2$. For $p = 3$, this is handled in [6] and [8] for the commutative and non-commutative set up, respectively, and for $p = 5, 7$ it is done in [9] in the commutative set up. The problem is open for the remaining values of $p$.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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